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Damping of Quantized Longitudinal Plasma Oscillations

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ABSTRACT

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Although the damping of longitudinal plasma oscillations has received considerable study for both classical plasmas and degenerate electron gases, expressions for damping in the degenerate systems at arbitrary temperatures have not been obtained. In this Report, the damping in the degenerate electron gas at arbitrary temperatures is evaluated within the random phase approximation (RPA) in the long-wavelength limit. In addition, exchange damping for a plasma slightly above the degeneracy temperature is evaluated and is shown to be comparable to the RPA damping for a wide range of parameters.

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I. INTRODUCTION

Damping of longitudinal plasma oscillations has been studied extensively for classical plasmas (Ref. 1) and, to a lesser extent, for degenerate plasmas at zero temperature (Refs. 2 and 3); however, expressions for damping in degenerate systems at arbitrary temperatures have not been obtained. Dubois, Gilinsky, and Kivelson (Ref. 4) have recently given a general treatment of damping, including correlation effects, but have performed explicit calculations only in the classical limit and at zero temperature. The damping of electron plasma oscillations near the degeneracy temperature has been considered by Morris (Ref. 5), but his calculation, based on the random phase approximation (RPA) discussed in Ref. 6, is incomplete for this region, since it neglects exchange effects. In this Report, the damping of a degenerate electron gas at arbitrary temperatures is evaluated within the RPA in the high-density, long-wavelength limit. In addition, exchange damping for a plasma near the degeneracy temperature is evaluated and shown to be comparable to the RPA damping coefficient for a wide range of parameters. The results obtained by the present authors in this part of the analysis have been published (Ref. 7).

II. THE RANDOM PHASE APPROXIMATION: PLASMA DAMPING AT ARBITRARY TEMPERATURES

The damping of electron plasma oscillations can be easily evaluated for arbitrary temperatures by making the random phase approximation: that is, all correlation and exchange effects are ignored. Physically, this assumption is valid for very-high-density degenerate electron gases or, alternatively, for low-density, high-temperature classical systems.

The RPA dispersion relation for quantized longitudinal plasma oscillations has been calculated many times (Ref. 2). One has the expression

$$1 = \frac{m}{\hbar q^2}\,\omega_p^2 \int \frac{\Delta F \, d^3k}{\left(-\omega + \dfrac{\hbar \mathbf{k}\cdot\mathbf{q}}{m} + \dfrac{\hbar q^2}{2m}\right)} \qquad (1)$$

where

$$\omega_p^2 = 4\pi \frac{Ne^2}{m} \tag{2}$$

$$\Delta F = F(\mathbf{k} + \mathbf{q}) - F(\mathbf{k}) \tag{3}$$

Here, F is the electron equilibrium distribution function, N is the electron density, e and m are the electron charge and mass, respectively, and $\hbar$ is the Planck constant divided by 2π. In addition, ω and $\mathbf{q}$ are the frequency and wave vector, respectively, of the plasma oscillation.

The damping of the plasma wave can be obtained in the usual manner by assuming that ω has a small positive imaginary part Γ, and by expanding the denominator of the integrand in Eq. 1. To lowest order in Γ, one obtains

$$1 = \omega_p^2 \frac{m}{\hbar q^3}\left[P \int \frac{\Delta F\, d^3 k}{D} + i\pi \int d^3k\, \Delta F \times \delta\left(\hbar \frac{\mathbf{k}\cdot\mathbf{q}}{mq} + \frac{\hbar q}{2m} - \frac{\omega_r}{q}\right) + i\frac{\Gamma}{q} P \int \frac{\Delta F}{D^2}\, d^3k\right] \tag{4}$$

where

$$D = \frac{\hbar\, \mathbf{k}\cdot\mathbf{q}}{mq} + \frac{\hbar q}{2m} - \frac{\omega_r}{q} \tag{5}$$

and

$$\omega = \omega_r + i\Gamma \tag{6}$$

In Eq. 4, P indicates that the principal value of the integral is to be taken. Equating the real and imaginary parts of Eq. 4 to zero, we find

$$1 = \frac{\omega_p^2}{\hbar}\frac{m}{q^3} P \int d^3k \frac{\Delta F}{D} \tag{7}$$

and

$$\Gamma = \frac{-\pi \int d^3k\, \Delta F\, \delta\left(\omega_r - \frac{\hbar\, \mathbf{k}\cdot\mathbf{q}}{m} - \frac{\hbar q^2}{2m}\right)}{P \int d^3k \frac{\Delta F}{D^2}} \tag{8}$$

The dispersion relation for ω_r, given by Eq. 7, is needed for calculation of the damping. An analysis of this quantity for arbitrary temperature is presented in Appendix A.

While the principal-value integrals in Eqs. 7 and 8 cannot be evaluated analytically, for the purposes considered here it is sufficient to make the long-wavelength approximation (Ref. 8). Near a temperature of 0°K, this amounts to assuming that $q << q_F$, where

$$q_F = \left(\frac{2m\,\epsilon_F}{\hbar^2}\right)^{1/2} = (3N\pi^2)^{2/3} = \frac{\omega_P}{\left(\frac{\epsilon_F}{m}\right)^{1/2}} \tag{9a}$$

is the Fermi wave number and ϵ_F is the Fermi energy. At temperatures above the degeneracy temperature, the cutoff wave number is taken to be the Debye wave number,

$$q_D = \frac{1}{\lambda_D} = \left(\frac{4\pi N e^2}{\kappa T}\right)^{1/2} \tag{9b}$$

Here, κ is the Boltzmann constant, and T is the absolute temperature. Within this approximation, the denominator of Eq. 8, obtained by differentiating Eq. 7, becomes

$$P \int d^3k \frac{\Delta F}{D^2} \simeq -\frac{2\hbar q^2}{m\,\omega_p^3} \tag{10}$$

Although the results obtained thus far can be applied to any type of quantum system, the most interesting physical example is the electron gas, for which the distribution function is

$$F = \frac{2}{N(2\pi)^3}\left[e^{\epsilon(k^2-\mu)} + 1\right]^{-1} \tag{11}$$

where

$$\epsilon = \frac{\hbar^2}{2m\,\kappa T} \tag{12}$$

and $\hbar^2\mu/2m$ is the chemical potential.

The evaluation of the integral in the numerator of Eq. 8 is trivial. For the damping, one obtains

$$\Gamma = \frac{-m^2\omega_p^2}{8\pi N \hbar^2 \epsilon q^3} \ln\left\{\frac{1 + e^{\epsilon[\mu - (Q-\alpha)^2]}}{1 + e^{\epsilon[\mu-(Q+\alpha)^2]}}\right\} \tag{13}$$

where

$$Q = \frac{m\,\omega_r}{\hbar q} \tag{14}$$

$$\alpha = q/2 \tag{15}$$

As can be seen, the damping is zero at $q = 0$. For temperatures near zero, we replace μ by its zero-temperature value[1], $\mu_0 = q_F^2$.

For small q, $(Q + \alpha)^2 > \mu_0$. As q becomes larger, we reach the value q_1 where

$$q_F^2 - (Q_1 - \alpha_1)^2 = 0 \tag{16}$$

For a still larger value of q, for example, q_2,

$$q_F^2 - (Q_2 + \alpha_2)^2 = 0$$

Near zero temperature, ϵ is very large. Thus, between q_1 and q_2, the numerator of the logarithm is becoming very

[1]This value can be found from the analysis in Appendix A.

large while the denominator is approaching 2. In the limit $T \rightarrow 0$, $\Gamma \rightarrow 0$ for $q < q_1$, but $\Gamma \rightarrow \infty$ at $q = q_{1+}$. For temperatures near zero, one would expect this value to be a good estimate for the cutoff. Using Eq. 16, one finds for this cutoff wave number

$$q_{max} \simeq q_F \left(-1 + \sqrt{1 + \frac{\hbar \omega_p}{\epsilon_F}} \right) \tag{17}$$

Since the condition for the validity of the RPA is $\hbar \omega_p << \epsilon_F$, we have

$$q_{max} \simeq \frac{1}{2} \frac{\hbar \omega_p}{\epsilon_F} q_F \tag{18}$$

At non-zero temperatures, Eq. 13 may be expanded in the long-wavelength approximation to obtain

$$\Gamma \simeq - \frac{m^2 \omega_p^3}{4\pi N \hbar^2 q^3 \epsilon} \exp \left\{ \epsilon \left[\mu - \left(\frac{m \omega_r}{\hbar q} \right)^2 - \frac{q^2}{4} \right] \right\} \times \sinh \frac{\hbar \omega_p}{2\kappa T} \tag{19}$$

One can see from Eq. 19 that specification of the chemical potential, a function of temperature and density, is sufficient to determine the damping. In Fig. 1 (with additional information in Table A-1) one finds curves of $\epsilon\mu$ vs T for different densities. By use of these curves and Eq. A-6, the damping was obtained for several values of N, q/q_F and T, and is given in Table 1. For temperatures below the degeneracy temperature, the damping is seen to be very small. As the temperature of the system is increased, exp $\epsilon\mu$ can be replaced by its high-temperature limit (Eq. A-10). At the degeneracy temperature, replacing exp $\epsilon\mu$ by the high-temperature limit would lead to an

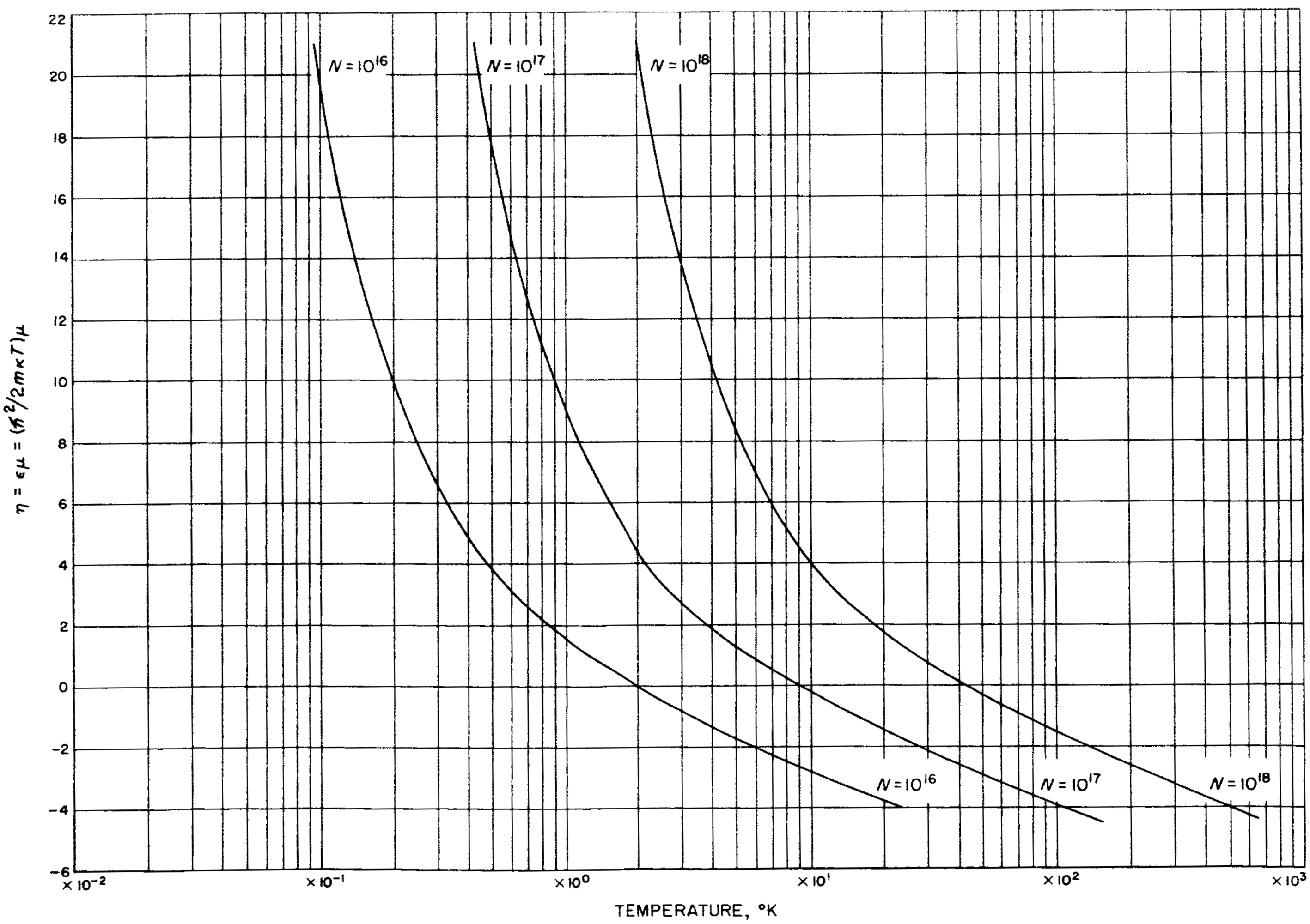

Fig. 1. Values of η as a function of temperature for $N = 10^{16}$, 10^{17}, and 10^{18}

underestimate of 35%. However, at $T = 2T_0$, the error is only 5%. Equation 19 then becomes

$$\Gamma = -\sqrt{\frac{\pi}{8}}\frac{\omega_p}{(q\,\lambda_D)^3}\left(\frac{2\,\kappa\,T}{\hbar\,\omega_p}\right)\sinh\frac{\hbar\,\omega_p}{2\,\kappa\,T}\exp\left(-\frac{1}{2}\frac{q_D^2}{q^2}\frac{\omega_r^2}{\omega_p^2}\right) \tag{20}$$

If ω_r is replaced by ω_p, this is the same expression as that obtained by Morris (Ref. 5). In the classical limit, one obtains the usual Landau damping. Morris has shown that, for temperatures slightly above the degeneracy temperature and densities below 10^{19} electrons/cm^3, the damping obtained from Eq. 20 can be much larger than the classical damping at the same temperature and density. Thus, in this region, the quantum aspect of the damping cannot be ignored.

Table 1. Values of Γ as a function of N, q/q_D, and T

T/T_0	$N = 10^{16}$		$N = 10^{18}$	
	Γ ($q/q_D = 0.1$)	Γ ($q/q_D = 0.2$)	Γ ($q/q_D = 0.1$)	Γ ($q/q_D = 0.2$)
0.5	1.41×10^{-43}	6.58×10^{-12}	7.29×10^{-43}	3.39×10^{-11}
1	2.22×10^{-22}	5.36×10^{-7}	1.18×10^{-21}	2.84×10^{-6}
3	1.27×10^{-8}	4.20×10^{-4}	6.30×10^{-8}	2.08×10^{-3}
10	3.52×10^{-4}	1.86×10^{-3}	1.22×10^{-3}	6.41×10^{-3}

III. EXCHANGE DAMPING

The RPA calculation in the previous Section is equivalent to the linearized Hartree approximation; that is, exchange effects are not included. However, von Roos (Ref. 8) has shown that, in the transition to the classical plasma, the leading quantum term in the dispersion relation is due to exchange. Thus, in a region where quantum effects are important, one might anticipate that exchange corrections to the damping could also be significant.

Following von Roos (Ref. 9), we find for the exchange contribution to the (complex) frequency

$$\omega_1 \chi = -\frac{\hbar \omega_p^2}{4m} \int \frac{du\, du'\, d^2 v_\perp\, d^2 v'_\perp\ \Delta F_0(v_\perp, u)\, \Delta F_0(v'_\perp, u')(u-u')^2}{\left[(u-u')^2 + (\mathbf{v}_\perp - \mathbf{v}'_\perp)^2\right]\left(u - \frac{\omega_0}{q} + \frac{\hbar q}{2m}\right)^2 \left(u' - \frac{\omega_0}{q} + \frac{\hbar q}{2m}\right)^2} = I \tag{22}$$

where

$$\chi = \int du\, d^2 v_\perp \frac{\Delta F_0 (v_\perp, u)}{\left(u - \frac{\omega_0}{q} + \frac{\hbar q}{2m}\right)^2} \tag{23}$$

In the expression above, ω_0 satisfies the RPA dispersion relation, Eq. 1. The velocity component u is parallel to $\mathbf{q}$ and $v_\perp$ is perpendicular to $\mathbf{q}$. Also, we write explicitly

$$\omega_0 = \omega_{0r} + i\Gamma_0 \qquad \Gamma_0 << \omega_{0r} \tag{24}$$

$$\omega_1 = \omega_{1r} + i\Gamma_1 \qquad \Gamma_1 << \omega_{1r} \tag{25}$$

We can now find Γ_1 from Eq. 22. One has

$$\Gamma_1 = -\frac{I_1 \chi_2}{\chi_1^2 - \chi_2^2} + \frac{I_2 \chi_1}{\chi_1^2 - \chi_2^2} \tag{26}$$

where I_1, I_2, χ_1, and χ_2 are the real and imaginary parts of I and χ, respectively.

Before performing the calculations, we shall make two assumptions. First, we make the long-wavelength approximation, $q\lambda_D << 1$. In addition, we assume that the temperature of the system is sufficiently high that a Maxwellian distribution can be used. This will be a good approximation for temperatures $T \geq 2\, T_0$, where T_0 is the degeneracy temperature ($T_0 = \epsilon_F/\kappa$).

The quantity χ may be calculated by differentiation of Eq. 1 with respect to q. Utilizing the relation

$$\lim_{\Gamma \to 0} \frac{1}{x \pm i\Gamma} = \frac{P}{x} \mp i\pi\,\delta(x) \tag{27}$$

one finds

$$\chi_1 = -\frac{2\hbar q^4}{m\,\omega_p^3} \tag{28}$$

$$\chi_2 = \frac{-2\hbar q^2}{m\,\lambda_D^2\,\omega_p^4}\,\Gamma_0 \tag{29}$$

$$\Gamma_0 = -\sqrt{\frac{\pi}{8}}\frac{\omega_{0r}}{(q\,\lambda_D)^3}\exp\left(-\frac{1}{2}\frac{q_D^2}{q^2}\frac{\omega_{0r}^2}{\omega_p^2}\right)\left(\frac{2\,\kappa\,T}{\hbar\,\omega_p}\right)\sinh\frac{h\,\omega_p}{2\,\kappa\,T} \tag{30}$$

In order to evaluate I, we write

$$I = I_1 + I_2 = I_{11} + I_{12} + I_2 \tag{31}$$

with

$$I_{11} = -\frac{\hbar\,\omega_p^2}{4m}P\int\frac{du\,du'\,d^2v_\perp\,d^2v'_\perp\,\Delta F_0(v_\perp,u)\,\Delta F_0(v'_\perp,u')\,(u-u')^2}{[(u-u')^2+(\mathbf{v}_\perp-\mathbf{v}'_\perp)^2]\left(u-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{2m}\right)^2\left(u'-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{2m}\right)^2} \tag{32}$$

$$I_{12} = \pi^2\frac{\hbar\,\omega_p^2}{4m}\int\frac{du\,du'\,d^2v_\perp\,d^2v'_\perp\,\Delta F_0(v_\perp,u)\,F_0(v'_\perp,u')\,(u-u')^2}{(u-u')^2+(\mathbf{v}_\perp-\mathbf{v}'_\perp)^2}$$
$$\times\frac{\partial}{\partial u}\delta\left(u-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{m}\right)\frac{\partial}{\partial u'}\delta\left(u'-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{m}\right) \tag{33}$$

$$I_2 = -\pi\frac{\hbar\,\omega_p^2}{m}\int\frac{du\,du'\,d^2v_\perp\,d^2v'_\perp\,\Delta F_0(v_\perp,u)\,\Delta F_0(v'_\perp,u')^2\,(u'-u)^2}{[(u-u')^2+(\mathbf{v}_\perp-\mathbf{v}'_\perp)^2]\left(u'-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{2m}\right)^3}$$
$$\times\,\delta\left(u-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{m}\right) \tag{34}$$

The quantity I_{11} has been evaluated by von Roos (Ref. 8), who obtained

$$I_{11} = \frac{7}{60}\,\frac{\hbar^3\,q^6}{2\lambda_D^2\,m^3\,\omega_p^2} \tag{35}$$

Some care must be exercised in the evaluation of I_{12} in order to prevent a spurious divergence. The difficulty may be circumvented by writing

$$-q\frac{\partial}{\partial\,\omega_{0r}}\delta\left(u-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{2m}\right) = \frac{\partial}{\partial u}\,\partial\left(u-\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{2m}\right) \tag{36}$$

and rewriting I_{12} as

$$I_{12} = \frac{\pi^2\,q^2\,\hbar\,\omega_p^2}{4m}\int d\,\omega'\,\delta(\omega'-\omega_{0r})\frac{\partial}{\partial\,\omega_{0r}}\frac{\partial}{\partial\,\omega'}$$
$$\times\int d^2v_\perp\,d^2v'_\perp\frac{\Delta F_0\left(v_\perp,\frac{\omega_{0r}}{q}-\frac{\hbar\,q}{2m}\right)\Delta F_0\left(v'_\perp,\frac{\omega'}{q}-\frac{\hbar\,q}{2m}\right)\left(\frac{\omega_{0r}-\omega'}{q}\right)^2}{\left(\frac{\omega_{0r}-\omega'}{q}\right)^2+(\mathbf{v}_\perp-\mathbf{v}'_\perp)^2} \tag{37}$$

Since we are using the Maxwellian distribution, we can write

$$\Delta F_0\left(v_\perp,\frac{\omega_{0r}}{q}-\frac{\hbar\,q}{2m}\right) = F_0(v_\perp)\left[f_0\left(\frac{\omega_{0r}}{q}+\frac{\hbar\,q}{2m}\right)-f_0\left(\frac{\omega_{0r}}{q}-\frac{\hbar\,q}{2m}\right)\right]$$
$$\Delta F_0 \simeq -2F_0(v_\perp)\sqrt{\frac{m}{2\pi\kappa T}}\exp\left[-\frac{1}{2}\left(\frac{q_D}{q}\frac{\omega_{0r}}{\omega_p}\right)^2\right]\sinh\frac{\hbar\,\omega_{0r}}{2\kappa T} \tag{38}$$

where the expansion of f_0 is consistent with the long-wavelength limit.

Also, one has

$$F_0(v_\perp) = \int du\, F_0(v_\perp, u) \tag{39}$$

Finally, I_{12} may be written

$$I_{12} = \frac{\pi\hbar\omega_p^2}{\kappa T} \exp - \left(\frac{q_D}{q}\frac{\omega_{0r}}{\omega_p}\right)^2 \sinh\frac{\hbar\omega_{0r}}{2\kappa T} G(\omega_{0r}, \omega')\Big|_{\omega' = \omega_{0r}} \tag{40}$$

where

$$G(\omega_{0r}, \omega') = P\int \frac{d^2 v_\perp\, d^2 v'_\perp\, F_0(v_\perp) F_0(v'_\perp)}{\left(\frac{\omega_{0r} - \omega'}{q}\right)^2 + (\mathbf{v}_\perp - \mathbf{v}'_\perp)^2} \tag{41}$$

Although this integral has not been evaluated in closed form, it is obviously bounded. The exponential factor appearing in I_{12} makes $I_{12} \ll I_{11}$, so that we can neglect this term.

The evaluation of I_2 must also be approximate. After making the long-wavelength expansion, I_2 may be reduced to

$$I_2 = A_1 \int \frac{d^2 v_\perp\, d^2 v'_\perp\, F_0(v_\perp) F_0(v'_\perp)}{\left(\frac{\omega_{0r}}{q}\right)^2 + (\mathbf{v}_\perp - \mathbf{v}'_\perp)^2} + A_2 \int \frac{d^2 v_\perp\, d^2 v'_\perp\, F_0(v_\perp) F_0(v'_\perp)}{\left[\left(\frac{\omega_{0r}}{q}\right)^2 + (\mathbf{v}_\perp - \mathbf{v}'_\perp)^2\right]^2} \tag{42}$$

where

$$A_1 = \frac{2\pi\hbar^2 q^3 \omega_p^2}{m^2 \omega_{0r}^2} \sqrt{\frac{m}{2\pi\kappa T}} \sinh\frac{\hbar\omega_{0r}}{2\kappa T} \exp\left[-\frac{1}{2}\left(\frac{q_D}{q}\frac{\omega_{0r}}{\omega_p}\right)^2\right] \tag{43}$$

$$A_2 = \frac{2\,\omega_{0r}^2}{q^2} A_1 \tag{44}$$

The integrals in Eq. 42 are evaluated in Appendix B. The final result for I_2 is

$$I_2 \simeq -\frac{6\hbar^3 q^8}{m^3 \omega_p^5} \Gamma_0 \tag{45}$$

Thus, substituting Eqs. 35 and 45 into Eq. 26, the final result for the exchange damping is

$$\Gamma_1 \simeq \frac{7}{60}\left(\frac{\hbar\omega_p}{2\kappa T}\right)^2 \Gamma_0 \tag{46}$$

and, for the total damping, one has

$$\Gamma = \Gamma_0\left[1 + \frac{7}{60}\left(\frac{\hbar\omega_p}{2\kappa T}\right)^2\right] \tag{47}$$

The exchange contribution to the total damping becomes larger than the RPA part when

$$\frac{\hbar\omega_p}{2\kappa T} > 3 \tag{48}$$

But, as seen from the table of parameters presented in Ref. 5, the exchange part is important exactly in the same region where other quantum effects are important: i.e., for temperatures slightly above the degeneracy temperature and densities below 10^{19} electrons/cm^3.

Even for $\hbar\,\omega_p/2\kappa T \ll 1$, an expansion of Eq. 19 reveals that

$$\Gamma = \Gamma_{classical}\left[1 + (1 + 0.7)\left(\frac{\hbar\,\omega_p}{2\kappa T}\right)\right] \tag{49}$$

where the contribution with the 0.7 coefficient results from exchange. Thus, even for this case, exchange cannot legitimately be ignored as compared with other quantum effects.

Finally, exchange can affect the damping through the appearance of ω_r in Γ_0. If we use the value of ω_r,

$$\omega_r = \omega_p\left\{1 + \left[\frac{3}{2} - \frac{7}{60}\left(\frac{\hbar\,\omega_p}{2\kappa T}\right)^2\right](q\lambda_D)^2\right\} \tag{50}$$

which includes the exchange effect[2], we obtain for Γ_0

$$\Gamma_0 \simeq -\sqrt{\frac{\pi}{8}}\,\frac{\omega_p}{(q\lambda_D)^3}\exp\left(\frac{-q_D^2}{2q^2}\right)\exp\left[-\frac{3}{2} + \frac{7}{60}\left(\frac{\hbar\,\omega_p}{2\kappa T}\right)^2\right] \tag{51}$$

Thus, the factor exp $-3/2$, which is incorrectly ignored in many treatments of Landau damping (Ref. 1), tends to be cancelled by exchange. Hence, again, the net effect of exchange is an enhancement of the damping.

[2]As pointed out by von Roos and Zmuidzinas in Ref. 9, a factor of two, resulting from a spin averaging, is missing from the exchange part of the expression given in Ref. 8.

APPENDIX A

The RPA Dispersion Relation at Arbitrary Temperatures

The dispersion relation in the random phase approximation, as given by Eq. 7 of Section II, can be rewritten as

$$1 = \omega_p^2 \int d^3k \frac{F(k)}{\left(\omega_r - \frac{\hbar \mathbf{k} \cdot \mathbf{q}}{m}\right)^2 - \left(\frac{\hbar q^2}{2m}\right)^2}$$

$$1 = \frac{\omega_p^2}{\omega_r^2}$$

$$\times \int \frac{d^3k\, F(k)}{\left(1 - \frac{\hbar \mathbf{k} \cdot \mathbf{q}}{m}\right)^2 \left[1 - \left(\frac{\hbar q}{2m}\right)^2 \frac{1}{\omega_r^2}\left(1 - \frac{\hbar \mathbf{k} \cdot \mathbf{q}}{m\, \omega_r}\right)^{-2}\right]} \qquad \text{(A-1)}$$

We now make the long-wavelength expansion of the denominator to obtain

$$\frac{\omega_r^2}{\omega_p^2} \simeq \int d^3k\, F(k) \left[1 - 2\frac{\hbar \mathbf{k} \cdot \mathbf{q}}{m\, \omega_p} + 3\left(\frac{\hbar \mathbf{k} \cdot \mathbf{q}}{m\, \omega_p}\right)^2 + \cdots\right]$$

$$= 1 + \frac{4\pi q^2 \hbar^2}{m^2 \omega_p^2} \int dk\, F(k)\, k^4 \qquad \text{(A-2)}$$

We introduce the $\mathcal{F}_\nu(\eta)$ integral according to the definition

$$\mathcal{F}_\nu(\eta) = \int \frac{dx\, x^\nu}{1 + e^{x-\eta}} \qquad \text{(A-3)}$$

where

$$x = \epsilon k^2 = \frac{\hbar^2 k^2}{2m\kappa T} \qquad \text{(A-4)}$$

and

$$\eta = \epsilon\mu \qquad \text{(A-5)}$$

For the Fermi distribution, the dispersion relation now becomes

$$\frac{\omega_r^2}{\omega_p^2} = 1 + \frac{\hbar^2 q^2 \mathcal{F}_{3/2}(\eta)}{2\pi^2 m^2 \eta_0^{5/2}} = 1 + 3\left(\frac{q}{q_F}\right)^2 \frac{\mathcal{F}_{3/2}(\eta)}{\eta_0^{5/2}} \qquad \text{(A-6)}$$

where

$$\eta_0 = (3N\pi^2)^{3/2} \epsilon \qquad \text{(A-7)}$$

The quantity η_0 is, itself, a function of temperature and density determined by the normalization condition; i.e.,

$$1 = \int d^3k\, F(k) = \frac{1}{2\pi^2 N}\left(\frac{2m\kappa T}{\hbar^2}\right)^{3/2} \mathcal{F}_{1/2}(\eta) \qquad \text{(A-8)}$$

Although $\mathcal{F}_{3/2}$ and $\mathcal{F}_{1/2}$ cannot be evaluated in closed form, tabulated values have been given by McDougall and Stoner (Ref. 10) for $-4 \leq \eta \leq 20$. In Table A-1, values of $\mathcal{F}_{1/2}(\eta)$, $\mathcal{F}_{3/2}(\eta)$, and T are given for a range of η for several densities. The tabulated values can be used in conjunction with Eq. A-8 to determine the temperature as a function of η for any other density.

In the two limiting cases of interest, Eq. A-6 reduces to the usual results. For the high-temperature limit,

$$\mathcal{F}_\nu(\eta) \simeq \Gamma(\nu + 1)\, e^\eta \qquad \eta < -4 \qquad \text{(A-9)}$$

so that

$$e^\eta = \frac{N}{2}(2\pi)^3 \left(\frac{\hbar}{2m\pi\kappa T}\right)^{3/2} \qquad \text{(A-10)}$$

Thus,

$$\frac{\omega_r^2}{\omega_p^2} \simeq 1 + 3\frac{q^2}{q_D^2} \qquad \text{(A-11)}$$

For the limit $T = 0$,

$$\mathcal{F}_\nu(\eta) = \frac{\eta_0^\nu}{\nu + 1} \qquad \eta > 20 \qquad \text{(A-12)}$$

so that

$$\frac{\omega_r^2}{\omega_p^2} = 1 + \frac{6}{5}\frac{q^2}{q_F^2} \qquad \text{(A-13)}$$

The intermediate region between these two limits is shown in Figs. A-1 and A-2, where ω/ω_p has been plotted as a function of q/q_F. Because we are in the long-wavelength limit, the curve has been cut off at $q/q_F = 0.2$. However, for $T > T_0$, where T_0 is the Fermi temperature, $q_D/q_F < 1$, and our curves are not really valid for $q/q_D = 0.2$. The point q_F/q_D has been indicated on the curves for $T > T_0$. The curve $T = 10\, T_0$ represents the high-temperature limit, Eq. A-11.

Table A-1. Values of $\mathcal{F}_{1/2}(\eta)$ and $\mathcal{F}_{3/2}(\eta)$ as functions of η for $N = 10^{16}$, 10^{17}, and 10^{18}

η	e^{η}	$\mathcal{F}_{1/2}(\eta)$	$\mathcal{F}_{3/2}(\eta)$	$N=10^{16}$	$N=10^{17}$	$N=10^{18}$
−4.0	1.8 −2	1.61 −2	2.43 −2	2.37 1	1.097 2	5.11 2
−3.0	5.0 −2	4.34 −2	6.56 −2	1.221 1	5.64 1	2.63 2
−2.0	1.36 −1	1.15 −1	1.75 −1	6.11 0	2.96 1	1.39 2
−1.0	3.68 −1	2.91 −1	4.61 −1	3.42 0	1.59 1	7.38 1
0	1.00 0	6.78 −1	1.15 0	1.96 0	9.09 0	4.22 1
1.0	2.72 0	1.40 0	2.66 0	1.20 0	5.58 0	2.60 1
2.0	7.39 0	2.50 0	5.54 0	8.18 −1	3.83 0	1.75 1
3.0	2.00 1	3.98 0	1.04 1	5.95 −1	2.81 0	1.302 1
4.0	5.50 1	5.77 0	1.76 1	4.71 −1	2.18 0	1.011 1
5.0	1.59 2	7.84 0	2.78 1	3.84 −1	1.80 0	8.31 0
6.0	4.03 2	1.014 1	4.13 1	3.25 −1	1.50 0	6.97 0
7.0	1.09 3	1.266 1	5.83 1	2.79 −1	1.29 0	5.98 0
8.0	2.98 3	1.54 1	7.94 1	2.43 −1	1.14 0	5.27 0
9.0	8.10 3	1.83 1	1.045 2	2.19 −1	1.02 0	4.69 0
10.0	2.20 4	2.13 1	1.343 2	1.98 −1	9.18 −1	4.22 0
11.0	5.99 4	2.46 1	1.69 2	1.80 −1	8.18 −1	3.73 0
12.0	1.63 5	2.80 1	2.08 2	1.64 −1	7.56 −1	3.53 0
13.0	4.42 5	3.15 1	2.53 2	1.51 −1	7.07 −1	3.26 0
14.0	1.20 6	3.51 1	3.03 2	1.40 −1	6.57 −1	3.04 0
15.0	3.27 6	3.89 1	3.58 2	1.31 −1	6.08 −1	2.84 0
16.0	8.88 6	4.29 1	4.19 2	1.23 −1	5.70 −1	2.65 0
17.0	2.41 7	4.69 1	4.87 2	1.16 −1	5.46 −1	2.50 0
18.0	6.56 7	5.11 1	5.60 2	1.10 −1	5.21 −1	2.38 0
19.0	1.78 8	5.54 1	6.40 2	1.04 −1	4.84 −1	2.26 0
20.0	4.85 8	5.98 1	7.26 2	9.84 −2	4.59 −1	2.13 0

Note: In columns 2 to 7, the digit spaced to the right of a 3-digit number represents the power of 10 by which to multiply the preceding 3-digit number.

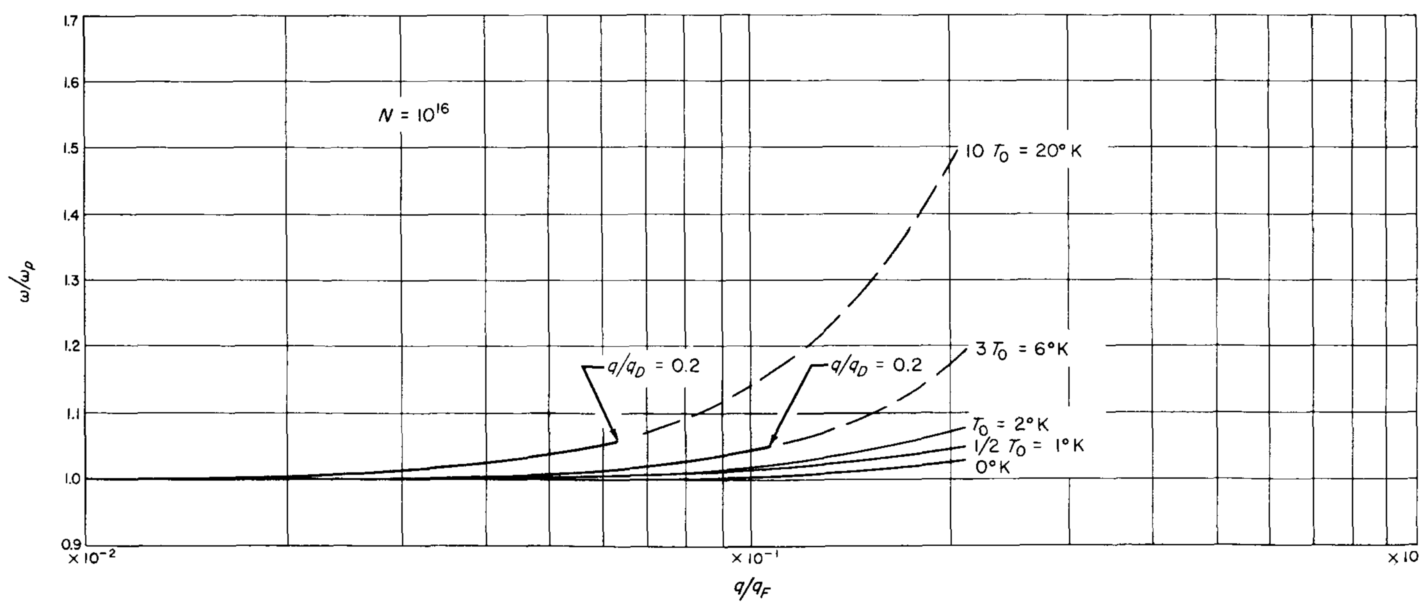

Fig. A-1. Frequency vs wavelength with temperature as a parameter, $N = 10^{16}$

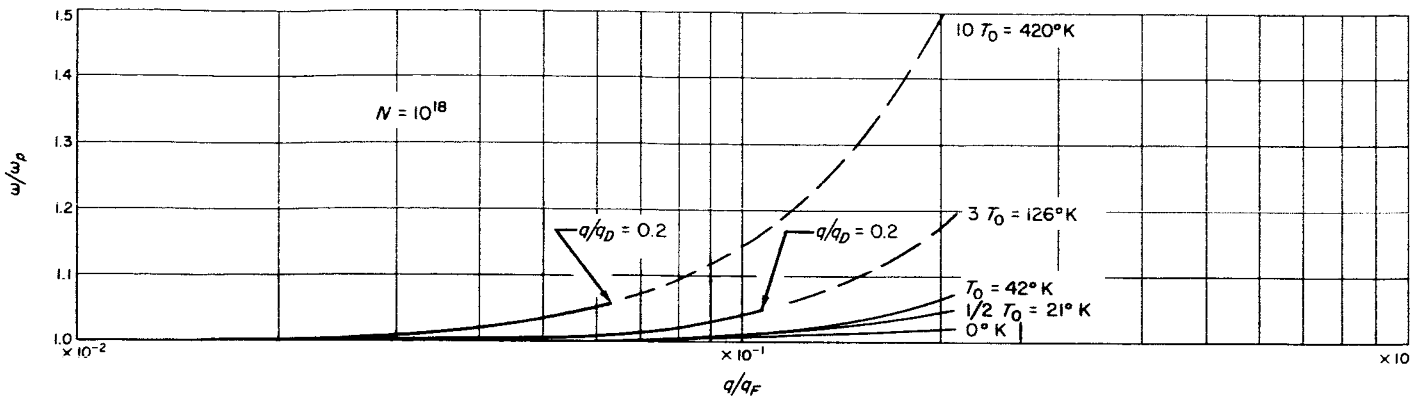

Fig. A-2. Frequency vs wavelength with temperature as a parameter, $N = 10^{18}$

APPENDIX B

Evaluation of Integrals in Equation 42

Consider the integral

$$J = \int d^2 v_\perp \, d^2 v'_\perp \, \frac{F(v_\perp) F(v'_\perp)}{\left(\frac{\omega_{0r}}{q}\right)^2 + (\mathbf{v}_\perp - \mathbf{v}'_\perp)^2} \tag{B-1}$$

Letting θ be the angle between $\mathbf{v}_\perp$ and $\mathbf{v}'_\perp$, we can write

$$J = \frac{\epsilon^2}{2\pi} \int_0^{2\pi} d\theta \int_0^\infty dz \int_0^\infty dz' \times \frac{e^{-\epsilon(z+z')}}{\left(\frac{\omega_{0r}}{q}\right)^2 + z + z' - 2\sqrt{zz'}\cos\theta} \tag{B-2}$$

where $\epsilon = m/\kappa T$. The θ integration can be performed to give

$$J = \epsilon^2 \int_0^\infty \frac{dz \, dz' \, e^{-\epsilon(z+z')}}{\left\{\left[\left(\frac{\omega_{0r}}{q}\right)^2 + z + z'\right]^2 - 4zz'\right\}^{1/2}} \tag{B-3}$$

Because ω_{0r}/q is large, we assume that

$$\left[\left(\frac{\omega_{0r}}{q}\right)^2 + z + z'\right]^2 \gg 4zz' \tag{B-4}$$

Then,

$$J = \epsilon^2 e^{\left(\frac{\omega_{0r}}{q}\right)^2} \int_0^\infty dz \int_{\left[\left(\frac{\omega_{0r}}{q}\right)^2 + z\right]\epsilon}^\infty dx \, \frac{e^{-x}}{x} \tag{B-5}$$

The area of integration is indicated in Fig. B-1. Changing the order of integration, we have

$$J = \epsilon^2 e^{\left(\frac{\omega_{0r}}{q}\right)^2} \int_{\epsilon\left(\frac{\omega_{0r}}{q}\right)^2}^\infty \frac{dx}{x} e^{-x} \int_0^{\frac{x}{\epsilon} - \left(\frac{\omega_{0r}}{q}\right)^2} dz$$

$$= \epsilon^2 \left[\frac{1}{\epsilon} - \left(\frac{\omega_{0r}}{q}\right)^2 e^{\left(\frac{\omega_{0r}}{q}\right)^2} \int_{\epsilon\left(\frac{\omega_{0r}}{q}\right)^2}^\infty \frac{dx}{x} e^{-x}\right] \tag{B-6}$$

For large lower limit, the integral can be expanded to give

$$J \simeq \left(\frac{q}{\omega_{0r}}\right)^2$$

The integral

$$H = \int d^2v_\perp \, d^2v'_\perp \frac{F(v_\perp)\,F(v'_\perp)}{\left[\left(\frac{\omega_{0r}}{q}\right)^2 + (\mathbf{v}_\perp - \mathbf{v}'_\perp)^2\right]^2} \tag{B-7}$$

can be evaluated in the same way to give

$$\mathrm{H} \simeq \left(\frac{q}{\omega_{0r}}\right)^4 \tag{B-8}$$

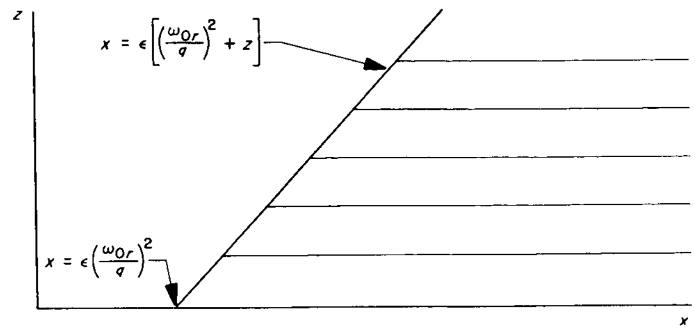

Fig. B-1. Area of integration for integral in Equation B-5

NOMENCLATURE

e	electron charge
F	electron equilibrium distribution function
$\hbar$	Planck constant divided by 2π
$\mathbf{k}$	particle momentum vector
m	electron mass
N	electron density
$\mathbf{q}$	wave vector of plasma oscillation
q_D	Debye wave number
q_F	Fermi wave number
RPA	random phase approximation
T	absolute temperature
u	electron-velocity component parallel to $\mathbf{q}$
$v_\perp$	electron-velocity component perpendicular to $\mathbf{q}$
Γ	small positive imaginary part of ω
δ	Dirac delta function
ϵ_F	Fermi energy
κ	Boltzmann constant
λ	wave length
ω	frequency of plasma oscillation

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